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Full rank interpolatory subdivision: A first encounter with the multivariate realm

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Abstract

We extend our previous work on interpolatory vector subdivision schemes to the multivariate case. As in the univariate case we show that the diagonal and off-diagonal elements of such a scheme have a significantly different structure and that under certain circumstances symmetry of the mask can increase the polynomial reproduction power of the subdivision scheme. Moreover, we briefly point out how tensor product constructions for vector subdivision schemes can be obtained.

Keywords: Multivariate vector subdivision; Full rank; Multivariate interpolatory scheme; Matrix tensor product scheme

1. Introduction

A *subdivision scheme* is an iterative process that generates curves or surfaces from given discrete data by refining this data on denser and denser grids. More specifically, starting with some (for simplicity scalar) initial data $c = (c_\alpha : \alpha \in \mathbb{Z}^s)$, defined on the integer grid, one iteratively computes a sequence $c^n := S_a^n c$, $n \in \mathbb{N}$, by repeated application of the stationary

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rule

$$(S_a c)_\alpha = \sum_{\beta \in \mathbb{Z}^s} a_{\alpha-2\beta}\, c_\beta, \quad \alpha \in \mathbb{Z}^s.$$

By assigning the sequences $S_a^n c$ to the denser and denser grids $2^{-n}\mathbb{Z}^s$, $n \in \mathbb{N}_0$, one can then establish a notion of convergence, not only to a continuous limit surface, but also in $L_p(\mathbb{R}^s)$, and most of the theory of stationary subdivision consists of reading off the *mask coefficients* a_α, $\alpha \in \mathbb{Z}^s$, whether convergence takes place and what properties the resulting limit surface has. For more than an introduction to stationary subdivision, we refer to [2,9].

A particular class of subdivision schemes are those that only refine the sequence c while keeping the "original data" in the sense that $(S_a c)_{2\alpha} = c_\alpha$, $\alpha \in \mathbb{Z}^s$. Such schemes are called *interpolatory* for obvious reasons and whenever such a scheme converges, the associated limit function is a cardinal interpolant to c, that is, its value at $\alpha \in \mathbb{Z}^s$ is c_α. Moreover, instead of scalar subdivision schemes as in the above explanation, one can also consider subdivision schemes that deal with *vector valued* data by using *matrix valued* masks. Such schemes play a role in the construction of multiwavelets [12] and of multichannel wavelets [1,5], i.e., wavelets for vector valued data. Interpolatory vector subdivision is the subject of this paper where we extend some of our previous results from the univariate situation to the slightly more intricate case of several variables.

Our main goals in this paper are the characterization of the symbols associated to multivariate full rank interpolatory schemes and the study of their polynomial reproduction properties, i.e., the question of when the subdivision scheme preserves all vector valued polynomials up to a certain degree. Moreover, since tensor product constructions are the simplest way to build multivariate objects from univariate ones, we also have a brief look at the question how to construct tensor products of *matrix valued* functions.

2. Vector subdivision schemes: Notation and background

We make use of the standard multiindex notation with $\alpha \in \mathbb{Z}^s$ or $\alpha \in \mathbb{N}_0^s$, respectively. By $|\alpha| = \|\alpha\|_1 = \sum |\alpha_j|$ we denote the *length* of α which will also indicate the order of partial derivative operators or the degree of polynomials. Moreover, $\alpha \le \beta$ for $\alpha, \beta \in \mathbb{N}_0^s$ if $\alpha_j \le \beta_j$, $j = 1, \ldots, s$, will stand for the partial ordering induced by componentwise comparison. For two multiindices $\alpha, \beta \in \mathbb{Z}^s$, we write $\alpha \cdot \beta = (\alpha_j \beta_j : j = 1, \ldots, s)$ for the componentwise, "Hadamard", product; the same notation also works for $z, z' \in \mathbb{C}^s$, of course. We will use the abbreviation ϵ_j for the unit multiindices in $\mathbb{Z}^s$, i.e., $(\epsilon_j)_k = \delta_{jk}$, $j, k = 1, \ldots, s$ and write $\mathbf{1}, -\mathbf{1}, \mathbf{0}$ for the vectors of proper size all whose components are 1, -1 or 0 respectively. Finally, we denote by $E = E_s := \{0, 1\}^s$ and $E' = E'_s := E_s \setminus \{0\}$, the vertices of the unit cube as well as E without the origin as the latter one often takes a special role. Since normally s is clear from the context, we will drop the subscript whenever possible to keep the notation as simple as we can.

We will also need some notation to indicate lower dimensional slices of coefficients. To that end, we will write $\mathbb{Z}_J^s := \{\alpha \in \mathbb{Z}^s : \alpha_j = 0,\ j \in J\}$, $J \subseteq \{1, \ldots, s\}$ with the abbreviation $\mathbb{Z}_j^s$ for $\mathbb{Z}_{\{j\}}^s$, the set of all multiindices whose jth component vanishes, $j \in \{1, \ldots, s\}$. The complement of $J \subset \{1, \ldots, s\}$ will be denoted by J^c.

For $r \in \mathbb{N}$ we write an $r \times r$ matrix $\boldsymbol{M} \in \mathbb{R}^{r\times r}$ as $\boldsymbol{M} = [m_{jk} : j, k = 1, \ldots, r]$ with infinity operator norm

$$|\boldsymbol{M}|_\infty = \max_{|\boldsymbol{y}|_\infty = 1} |\boldsymbol{M}\boldsymbol{y}|_\infty,$$

where $|\cdot|_\infty$ stands for the supremum norm of a vector in $\mathbb{R}^r$, and denote by $\ell^{r\times r}(\mathbb{Z}^s)$ the space of all $r \times r$-matrix valued sequences

$$\boldsymbol{A} = \left(\boldsymbol{A}_\alpha \in \mathbb{R}^{r\times r} : \alpha \in \mathbb{Z}^s\right).$$

For notational simplicity we write $\ell^r(\mathbb{Z}^s)$ for $\ell^{r\times 1}(\mathbb{Z}^s)$ and denote vector sequences by lowercase letters like $\boldsymbol{c} = (\boldsymbol{c}_\alpha \in \mathbb{R}^r : \alpha \in \mathbb{Z}^s)$. Moreover, we denote by $\ell_\infty^{r\times r}(\mathbb{Z}^s)$ the Banach space of all $r\times r$-matrix valued sequences with uniformly bounded operator norm, defined for $\boldsymbol{A} \in \ell_\infty^{r\times r}(\mathbb{Z}^s)$ by

$$\|\boldsymbol{A}\|_\infty := \sup_{\alpha\in\mathbb{Z}^s} |\boldsymbol{A}_\alpha|_\infty . \tag{1}$$

Furthermore, we denote by $\ell_0^{r\times r}(\mathbb{Z}^s)$ the space of finitely supported sequences, hence $\boldsymbol{A} \in \ell_0^{r\times r}(\mathbb{Z}^s)$ means that $\boldsymbol{A}_\alpha = \boldsymbol{0}$ for $\alpha \notin [-N, N]^s$ for some suitable $N \in \mathbb{N}$.

Moreover, $C_u^{r\times r}(\mathbb{R}^s)$ will denote the Banach space of all uniformly continuous and uniformly bounded $r \times r$-matrix valued functions on $\mathbb{R}^s$ with the norm

$$\|\boldsymbol{F}\|_\infty := \sup_{x\in\mathbb{R}^s} |\boldsymbol{F}(x)|_\infty.$$

For two matrix sequences as well as for a matrix function and a matrix sequence we introduce the *convolution* "$*$" defined, respectively, as

$$(\boldsymbol{A} * \boldsymbol{B})_\alpha := \sum_{\beta\in\mathbb{Z}^s} \boldsymbol{A}_{\alpha-\beta} \boldsymbol{B}_\beta, \qquad (\boldsymbol{F} * \boldsymbol{B})(x) := \sum_{\beta\in\mathbb{Z}^s} \boldsymbol{F}(x-\beta) \boldsymbol{B}_\beta.$$

We recall that a *vector subdivision operator*, based on a finitely supported *matrix mask* $\boldsymbol{A} \in \ell_0^{r\times r}(\mathbb{Z}^s)$, is a linear operator acting on a vector sequence $\boldsymbol{c} \in \ell^r(\mathbb{Z}^s)$ as

$$S_{\boldsymbol{A}}\boldsymbol{c} = \left(\sum_{\beta\in\mathbb{Z}^s} \boldsymbol{A}_{\alpha-2\beta} \boldsymbol{c}_\beta : \alpha \in \mathbb{Z}^s\right).$$

A *subdivision scheme* consists of iterating the subdivision operator on an initial vector sequence $\boldsymbol{c}^0 = \boldsymbol{c} \in \ell_\infty^r(\mathbb{Z}^s)$, namely:

$$\begin{aligned} &\boldsymbol{c}^0 := \boldsymbol{c} \\ &\boldsymbol{c}^n := S_{\boldsymbol{A}}\boldsymbol{c}^{n-1} = S_{\boldsymbol{A}}^n\boldsymbol{c}, \qquad n \geq 1. \end{aligned}$$

The subdivision scheme is called L_∞-*convergent* if for any $\boldsymbol{c} \in \ell_\infty^r(\mathbb{Z}^s)$ there exists a uniformly continuous vector valued *limit function* $\boldsymbol{f}_{\boldsymbol{c}} \in C^r(\mathbb{R}^s)$ such that

$$\lim_{n\to\infty} \sup_{\alpha\in\mathbb{Z}^s} \left|\left(S_{\boldsymbol{A}}^n\boldsymbol{c}\right)_\alpha - \boldsymbol{f}_{\boldsymbol{c}}\left(2^{-n}\alpha\right)\right|_\infty = 0. \tag{2}$$

An equivalent description of convergence is the existence of the *basic limit function* as uniform limit of the matrix sequence $S_{\boldsymbol{A}}^n\delta\boldsymbol{I}$ (where δ is the scalar sequence $\delta_0 = 1, \delta_\alpha = 0, \alpha \neq 0$), that is, the existence of an uniformly continuous matrix valued function $\boldsymbol{F}$ such that

$$\lim_{n\to\infty} \sup_{\alpha\in\mathbb{Z}^s} \left|\left(S_{\boldsymbol{A}}^n\delta\boldsymbol{I}\right)_\alpha - \boldsymbol{F}\left(2^{-n}\alpha\right)\right|_\infty = 0. \tag{3}$$

In fact, in the case of convergence we have that

$$\boldsymbol{f}_c = \boldsymbol{F} * \boldsymbol{c} = \sum_{\alpha\in\mathbb{Z}^s} \boldsymbol{F}(\cdot - \alpha)\, \boldsymbol{c}_\alpha.$$

The basic limit function is *refinable* with respect to $\boldsymbol{A}$, which means that it satisfies the functional equation

$$\boldsymbol{F} = (\boldsymbol{F} * \boldsymbol{A})\,(2\cdot) = \sum_{\alpha \in \mathbb{Z}^s} \boldsymbol{F}\,(2\cdot -\alpha)\,\boldsymbol{A}_\alpha. \tag{4}$$

A useful tool for subdivision analysis is the *symbol*

$$\boldsymbol{A}^*(z) = \sum_{\alpha \in \mathbb{Z}^s} \boldsymbol{A}_\alpha\, z^\alpha, \qquad z \in \mathbb{C}_\times^s, \qquad \mathbb{C}_\times := \mathbb{C} \setminus \{0\},$$

associated to the mask $\boldsymbol{A}$. Since the mask is always supposed to be finitely supported, the symbol is a Laurent polynomial.

The *subsymbols* of the mask $\boldsymbol{A}$ are the matrix valued Laurent polynomials

$$\boldsymbol{A}_\eta^*(z) = \sum_{\alpha \in \mathbb{Z}^s} \boldsymbol{A}_{\eta+2\alpha}\, z^\alpha, \quad \eta \in E, z \in \mathbb{C}_\times^s,$$

and they satisfy

$$\boldsymbol{A}^*(z) = \sum_{\eta \in E} z^\eta\, \boldsymbol{A}_\eta^*\left(z^2\right). \tag{5}$$

The *rank* of the mask $\boldsymbol{A}$ or of the associated subdivision scheme $S_{\boldsymbol{A}}$ is the number $R(\boldsymbol{A}) := \dim \mathcal{E}_{\boldsymbol{A}}$ where $\mathcal{E}_{\boldsymbol{A}}$ is the joint 1-eigenspace for the matrices $\boldsymbol{A}_\eta^*(1)$, $\eta \in E$, namely,

$$\mathcal{E}_{\boldsymbol{A}} := \left\{\boldsymbol{y} \in \mathbb{R}^r : \boldsymbol{A}_\eta^*(1)\,\boldsymbol{y} = \boldsymbol{y},\ \eta \in E\right\}. \tag{6}$$

For convergent schemes we have that $1 \le R(\boldsymbol{A}) \le r$, cf. [13], and a subdivision scheme with mask $\boldsymbol{A}$ is said to be *of full rank* if $R(\boldsymbol{A}) = r$. The *rank* of a matrix function $\boldsymbol{F} \in C^{r\times r}\,(\mathbb{R}^s)$ is likewise defined as

$$R(\boldsymbol{F}) := r - \dim\left\{\boldsymbol{y} \in \mathbb{R}^r : \boldsymbol{y}^T \boldsymbol{F}(x) = 0, x \in \mathbb{R}^s\right\}.$$

Also recall that the Kronecker product of two matrices $\boldsymbol{M} \in \mathbb{R}^{k\times\ell}$ and $\boldsymbol{N} \in \mathbb{R}^{r\times s}$ is defined as the block matrix

$$\boldsymbol{M} \otimes \boldsymbol{N} = \begin{bmatrix} m_{11}\boldsymbol{N} & \dots & m_{1\ell}\boldsymbol{N} \\ \vdots & \ddots & \vdots \\ m_{k1}\boldsymbol{N} & \dots & m_{k\ell}\boldsymbol{N} \end{bmatrix} \in \mathbb{R}^{kr\times\ell s}.$$

3. Full rank subdivision schemes

As a reasonable minimal requirement, vector interpolatory subdivision operators should at least preserve *constant vector data*, which means that whenever $\boldsymbol{c} \in \ell_\infty^r(\mathbb{Z}^s)$ is a constant sequence, such that $\boldsymbol{c}_\alpha = \boldsymbol{y}$, $\alpha \in \mathbb{Z}^s$, for some $\boldsymbol{y} \in \mathbb{R}^r$, then $S_{\boldsymbol{A}}\boldsymbol{c}$ should yield the same constant sequence, that is $(S_{\boldsymbol{A}}\boldsymbol{c})_\alpha = \boldsymbol{y}$, $\alpha \in \mathbb{Z}^s$. Preservation of constant sequences is equivalent to

$$\boldsymbol{A}_\eta^*(1) = \sum_{\alpha \in \mathbb{Z}} \boldsymbol{A}_{\eta+2\alpha} = \boldsymbol{I}, \quad \eta \in E, \tag{7}$$

implying that the subdivision operator is *of full rank* since, in that case, it obviously follows that $\mathcal{E}_{\boldsymbol{A}} = \mathbb{R}^r$. Note that the conditions (7) are equivalent to

$$\boldsymbol{A}^*(1) = 2^s \boldsymbol{I}, \qquad \boldsymbol{A}^*\left((-1)^\eta\right) = \boldsymbol{0}, \quad \eta \in E'. \tag{8}$$

As shown e.g. in [3] the conditions (8) imply that the symbol can always be factorized and a so-called difference scheme can be defined. To explain this fact in more detail, we introduce the difference operator and its interaction with the vector subdivision scheme. Denote by ∇_ℓ, $\ell = 1, \dots, s$, the ℓth *partial backwards difference operator* acting on vector sequences $c \in \ell^r(\mathbb{Z}^s)$ by means of

$$(\nabla_\ell \boldsymbol{c})_\alpha := \boldsymbol{c}_{\alpha-\epsilon_\ell} - \boldsymbol{c}_\alpha, \quad \alpha \in \mathbb{Z}^s, \ell = 1, \dots, s.$$

We prefer to choose the operator in that particular fashion as its symbol then takes the convenient form

$$(\nabla_\ell \boldsymbol{c})^*(z) = (z_\ell - 1)\, \boldsymbol{c}^*(z), \qquad \text{i.e. } \nabla_\ell^*(z) = z_\ell - 1.$$

The *backwards difference operator* $\nabla : \ell^r(\mathbb{Z}^s) \to \ell^{rs}(\mathbb{Z}^s)$ is then the gradient-like object defined as

$$\nabla = \begin{bmatrix} \nabla_1 \\ \nabla_2 \\ \vdots \\ \nabla_s \end{bmatrix}, \qquad \nabla^*(z) = \left[z_j - 1 : j = 1, \dots, s\right].$$

For $k > 1$, up to a repetition of rows due to the commuting of the partial difference operators, we can define the power of the backwards difference operator or, equivalently, a total difference operator of order k, $\nabla^k : \ell^r(\mathbb{Z}^s) \to \ell^{rs^k}(\mathbb{Z}^s)$ in its block representation as

$$\nabla^k := [\nabla^\alpha : |\alpha| = k], \qquad \nabla^\alpha := \nabla_1^{\alpha_1} \cdots \nabla_s^{\alpha_s}. \tag{9}$$

We continue with a proposition discussing the factorization property.

Proposition 1 ([3,15]). *Assume that $\boldsymbol{A} \in \ell_0^{r\times r}(\mathbb{Z}^s)$ is full rank. Then there exists a mask $\boldsymbol{B} \in \ell_0^{rs\times rs}(\mathbb{Z}^s)$ such that*

$$\nabla S_{\boldsymbol{A}} = S_{\boldsymbol{B}} \nabla, \tag{10}$$

and $\boldsymbol{B}$ is of full rank again.

Eq. (10) is the aforementioned *factorization property*. Indeed, in the univariate case this is equivalent to $\boldsymbol{A}^*(z)$ having a factor of the form $z + 1$. In higher dimensions this property corresponds to the components of $\boldsymbol{A}^*(z)$ belonging to a certain *quotient ideal*, cf. [14].

Polynomial reproduction for multivariate full rank schemes has been investigated in [6] where conditions on the subdivision mask are given. To recall such results we introduce the (scalar) monomial sequences

$$\pi_\beta := \left(\pi_\beta(\alpha) = \alpha^\beta, \alpha \in \mathbb{Z}^s\right), \quad \beta \in \mathbb{N}_0^s,$$

and define the spaces of vector valued polynomial sequences of total degree at most k

$$\Pi_k^r(\mathbb{Z}^s) := \left\{ \boldsymbol{p} \in \ell^r(\mathbb{Z}^s) : \boldsymbol{p} = \sum_{|\beta|\le k} \boldsymbol{p}_\beta\, \pi_\beta,\ \boldsymbol{p}_\beta \in \mathbb{R}^r, |\beta| \le k \right\}$$

as well as the space of polynomial vector sequences

$$\Pi^r\left(\mathbb{Z}^s\right) := \bigcup_{k\geq 0} \Pi_k^r\left(\mathbb{Z}^s\right).$$

We recall from [6] that a matrix valued function $\boldsymbol{F}$ is said to be *boundedly independent* if

$$\sum_{\alpha\in\mathbb{Z}^s} \eta^\alpha \boldsymbol{F}(\cdot-\alpha)\boldsymbol{y} \neq \boldsymbol{0}, \quad \eta\in\{-1,1\}^s, \boldsymbol{y}\in\mathbb{R}^r\setminus\{\boldsymbol{0}\}. \tag{11}$$

Bounded independence is a somewhat weaker concept than stability and in some sense a minimal condition to relate properties of the mask and the associated refinable function, see again [6].

Theorem 2 (*[6]*). *Let $\boldsymbol{F}$ be a boundedly independent matrix function, refinable with respect to the mask $\boldsymbol{A}\in\ell_0^{r\times r}(\mathbb{Z}^s)$ and with the additional property that*

$$\boldsymbol{F}_0 := \sum_{\alpha\in\mathbb{Z}^s} \boldsymbol{F}(\alpha)$$

is an invertible matrix. Then $\Pi_k^r(\mathbb{Z}^s) \subseteq \mathcal{S}(\boldsymbol{F}) := \{\boldsymbol{F}*\boldsymbol{c} : \boldsymbol{c}\in\ell^r(\mathbb{Z}^s)\}$ if and only if

1. $\boldsymbol{A}^*(1) = 2^s\boldsymbol{I}$,
2. $D^\beta\boldsymbol{A}^*((-1)^\eta) = 0$ *for* $|\beta|\leq k$ *and* $\eta\in E'$.

4. Interpolatory full rank subdivision schemes

As pointed out in [4], full rank schemes appear most naturally in the context of *interpolatory* vector subdivision schemes which are characterized by the property that

$$(S_{\boldsymbol{A}}\boldsymbol{c})_{2\alpha} = \boldsymbol{c}_\alpha, \quad \alpha\in\mathbb{Z}^s, \tag{12}$$

or, equivalently,

$$\boldsymbol{A}_{2\alpha} = \delta_{\alpha,0}\boldsymbol{I}, \quad \alpha\in\mathbb{Z}^s.$$

We can also describe these properties in terms of the symbol $\boldsymbol{A}^*(z)$ as done in the following propositions.

Proposition 3. *A subdivision scheme $S_{\boldsymbol{A}}$ is interpolatory if and only if*

$$\sum_{\eta\in E} \boldsymbol{A}^*\left((-1)^\eta\cdot z\right) = 2^s\boldsymbol{I}, \quad z\in\mathbb{C}_\times^s. \tag{13}$$

Proof. The proof is based on the identity

$$\sum_{\eta\in E} \boldsymbol{A}^*\left((-1)^\eta\cdot z\right) = 2^s \sum_{\alpha\in\mathbb{Z}^s} \boldsymbol{A}_{2\alpha} z^{2\alpha}, \tag{14}$$

which is easily verified by direct computations. Since $S_{\boldsymbol{A}}$ is interpolatory if and only if $\boldsymbol{A}_{2\alpha} = \delta_{\alpha,0}\boldsymbol{I}$, a comparison of coefficients in (14) shows that also (13) is equivalent to the scheme being interpolatory. □

Proposition 4. *An interpolatory scheme $S_{\boldsymbol{A}}$ is of full rank if and only if*

$$\boldsymbol{A}^*(1) = 2^s\boldsymbol{I} \quad \text{and} \quad \boldsymbol{A}_\eta^*(1) = \boldsymbol{A}_\theta^*(1), \quad \eta,\theta\in E'. \tag{15}$$

Proof. If $S_{\boldsymbol{A}}$ is of full rank then $\boldsymbol{A}_\eta^*(1) = \boldsymbol{I}$, $\eta \in E$, and so (15) holds by definition. Conversely, it follows from (12) that $\boldsymbol{A}_0^*(1) = \boldsymbol{I}$, and setting $z = 1$ in (5) we find that

$$\sum_{\eta \in E'} \boldsymbol{A}_\eta^*(1) = \boldsymbol{A}^*(1) - \boldsymbol{A}_0^*(1) = \left(2^s - 1\right) \boldsymbol{I},$$

which implies together with (15) that $\boldsymbol{A}_\eta^*(1) = \boldsymbol{I}$, $\eta \in E'$, proving that the scheme is full rank. $\square$

Proposition 5. *Suppose that $\boldsymbol{A} \in \ell_0^{r\times r}(\mathbb{Z}^s)$ is an interpolatory full rank mask which admits a convergent subdivision scheme. Then the basic limit function $\boldsymbol{F}$ is a cardinal partition of the identity, i.e.*

$$\boldsymbol{F}(\alpha) = \delta_\alpha \boldsymbol{I} \quad \textit{and} \quad \sum_{\alpha \in \mathbb{Z}^s} \boldsymbol{F}(\cdot - \alpha) = \boldsymbol{I}. \tag{16}$$

Proof. For the initial matrix sequence $\delta \boldsymbol{I}$, the interpolation property (12) yields for $\alpha \in \mathbb{Z}^s$ that

$$\left(S_{\boldsymbol{A}}^n \delta \boldsymbol{I}\right)_{2^n\alpha} = \delta_\alpha \boldsymbol{I}, \quad \alpha \in \mathbb{Z}^s.$$

Taking the limit for $n \to \infty$, this gives $\boldsymbol{F}(\alpha) = \delta_\alpha \boldsymbol{I}, \alpha \in \mathbb{Z}^s$. For the partition of identity property, we just note that $\boldsymbol{A}$ being of full rank implies $S_{\boldsymbol{A}}^n \boldsymbol{I} = \boldsymbol{I}$, $n \geq 1$ and that the left-hand side of this identity converges to $\boldsymbol{F} * \boldsymbol{I}$. $\square$

We immediately see from (8) and (13) that the diagonal entries of a full rank interpolatory symbol $\boldsymbol{A}^*(z)$ must satisfy, for $j = 1, \ldots, r$,

$$2^s\, \delta_\eta = a_{jj}^*\left((-1)^\eta\right), \quad \eta \in E, \tag{17}$$

$$2^s = \sum_{\eta \in E} a_{jj}^*\left((-1)^\eta \cdot z\right), \quad z \in \mathbb{C}_\times^s, \tag{18}$$

thus, they must be symbols of *scalar interpolatory schemes* which reproduce at least constants.

The requirements for the off-diagonal elements of a full rank interpolatory subdivision scheme are slightly more intricate than those for the diagonal elements and will be described as follows.

Proposition 6. *The off-diagonal elements $a_{\ell m}^*(z)$, $\ell \neq m$, of the symbol $\boldsymbol{A}^*(z)$ of a full rank interpolatory matrix subdivision scheme can be written as*

$$a_{\ell m}^*(z) = \sum_{j=1}^s \left(z_j^2 - 1\right)\left(b_{\ell m}^j\right)^*(z), \quad z \in \mathbb{C}_\times^s, \tag{19}$$

where

$$\sum_{\alpha \in \mathbb{Z}_j^s} \left(b_{\ell m}^j\right)_{2(\alpha + k\epsilon_j)} = 0, \quad j = 1, \ldots, s, k \in \mathbb{Z}. \tag{20}$$

Proof. Due to (8), any off-diagonal element $a_{\ell,m}^*(z)$, $\ell \neq m$, of $\boldsymbol{A}^*(z)$, can be written as

$$a_{\ell m}^*(z) = \sum_{j=1}^s \left(z_j^2 - 1\right)\left(b_{\ell m}^j\right)^*(z), \quad z \in \mathbb{C}_\times^s. \tag{21}$$

Next, we consider the interpolation property (13) to obtain the additional condition

$$
\begin{aligned}
0 &= \sum_{\eta \in E} a^*_{\ell m}\left((-z)^\eta\right) = \sum_{\eta \in E}\sum_{j=1}^{s}\left(z_j^2 - 1\right)\left(b^j_{\ell m}\right)^*\left((-1)^\eta \cdot z\right) \\
&= \sum_{j=1}^{s}\left(z_j^2 - 1\right)\sum_{\eta \in E}\left(b^j_{\ell m}\right)^*\left((-1)^\eta \cdot z\right).
\end{aligned} \tag{22}
$$

With the abbreviation

$$
\boldsymbol{p}(z) := \left[p_j(z) : j = 1, \ldots, s\right] := \left[\sum_{\eta \in E}\left(b^j_{\ell m}\right)^*\left((-1)^\eta \cdot z\right) : j = 1, \ldots, s\right],
$$

and the s-tuple or "vector" $\left[z^2 - 1\right] := \left[z_j^2 - 1 : j = 1, \ldots, s\right]$, Eq. (22) can be written more compactly as

$$
\left[z^2 - 1\right]^T \boldsymbol{p}(z) = 0,
$$

hence, $\boldsymbol{p}(z)$ is a *syzygy* of $\left[z^2 - 1\right]$. Recall, e.g. from [7], that a syzygy for a "vector" $\boldsymbol{p}$ of (Laurent) polynomials can naively be seen as a "linear dependency relation" in the ring of (Laurent) polynomials, that is, a vector $\boldsymbol{q}$ of (Laurent) polynomials such that $\boldsymbol{p}^T\boldsymbol{q} = 0$. Strictly speaking, of course all this takes place in a module generated by the ring of (Laurent) polynomials, cf. [8,10].

Fix $j \in \{1, \ldots, s\}$ and write $z = \left(z_j, \widehat{z}\right)$, where $\widehat{z} = z \setminus \left\{z_j\right\}$; in the same sense we will also use $\widehat{\alpha}$ and $\widehat{\eta}$ etc., thus rearranging the variables and the associated indices suitably. For any $\widehat{\eta} \in \widehat{E} := E_{s-1}$ we now have that

$$
\left[\left(z_j, (-1)^{\widehat{\eta}}\right)^2 - 1\right]^T \boldsymbol{p}\left(z_j, (-1)^{\widehat{\eta}}\right) = 0,
$$

hence

$$
\begin{aligned}
0 &= p_j\left(z_j, (-1)^{\widehat{\eta}}\right) \\
&= \sum_{\theta \in E}\left(b^j_{\ell m}\right)^*\left((-1)^{\theta_j} z_j, (-1)^{\widehat{\theta}+\widehat{\eta}}\right) \\
&= \sum_{\widehat{\theta} \in \widehat{E}}\left(b^j_{\ell m}\right)^*\left(z_j, (-1)^{\widehat{\theta}+\widehat{\eta}}\right) + \left(b^j_{\ell m}\right)^*\left(-z_j, (-1)^{\widehat{\theta}+\widehat{\eta}}\right) \\
&= \sum_{\widehat{\theta} \in \widehat{E}}\left(b^j_{\ell m}\right)^*\left(z_j, (-1)^{\widehat{\theta}}\right) + \left(b^j_{\ell m}\right)^*\left(-z_j, (-1)^{\widehat{\theta}}\right)
\end{aligned}
$$

since in the summation over $\widehat{\theta}$ every combination of odd and even values in $\widehat{\theta} + \widehat{\eta}$ appear exactly once, and this independently of η. We fix ℓ and m and write

$$
\left(b^j_{\ell m}\right)^*(z) = \sum_{\alpha \in \mathbb{Z}^s} b^j_\alpha\, z^\alpha,
$$

to obtain

$$
0 = \sum_{\widehat{\theta} \in \widehat{E}}\sum_{\alpha_j \in \mathbb{Z}}\sum_{\widehat{\alpha} \in \mathbb{Z}^{s-1}} b^j_\alpha z_j^{\alpha_j}\left(1 + (-1)^{\alpha_j}\right)(-1)^{\widehat{\alpha}^T\widehat{\theta}}
$$

$$= 2 \sum_{\alpha_j \in 2\mathbb{Z}} \sum_{\widehat{\alpha} \in \mathbb{Z}^{s-1}} b^j_\alpha z_j^{\alpha_j} \sum_{\widehat{\theta} \in \widehat{E}} (-1)^{\widehat{\alpha}^T \widehat{\theta}} = 2^s \sum_{\alpha \in 2\mathbb{Z}^s} b^j_\alpha z_j^{\alpha_j}$$

since

$$\sum_{\widehat{\theta} \in \widehat{E}} (-1)^{\widehat{\alpha}^T \widehat{\theta}} = \begin{cases} 2^{s-1}, & \widehat{\alpha} \in 2\mathbb{Z}^{s-1}, \\ 0, & \widehat{\alpha} \notin 2\mathbb{Z}^{s-1}. \end{cases}$$

Consequently,

$$0 = \sum_{\alpha \in \mathbb{Z}^s_j} b^j_{2(\alpha + k\epsilon_j)}, \quad k \in \mathbb{Z},$$

and this has to hold for any $j, \ell, m \in \{1, \dots, s\}$, $\ell \neq m$. $\square$

5. Symmetry and polynomial reproduction

The analysis of polynomial reproduction for interpolatory full rank masks is based on Theorem 2 under the additional assumption that the subdivision mask $\boldsymbol{A}$ satisfies the interpolatory condition $\boldsymbol{A}_{2\alpha} = \delta_\alpha \boldsymbol{I}$, $\alpha \in \mathbb{Z}^s$.

Obviously, we have for $\beta = 0$ that

$$D^\beta \boldsymbol{A}^* \left((-1)^\eta \right) = \boldsymbol{A}^* \left((-1)^\eta \right) = 0, \quad \eta \in E',$$

so that the polynomial reproduction of degree 0, i.e., the reproduction of constants, is guaranteed as a minimal requirement for the full rank scheme to be convergent. Also, the cardinality of the function $\boldsymbol{F}$ ensures that $\boldsymbol{F}_0 := \sum_{\alpha \in \mathbb{Z}^s} \boldsymbol{F}(\alpha) = \boldsymbol{I}$ is an invertible matrix.

To investigate the polynomial reproduction of higher order, a necessary condition for the limit function to be differentiable, we need an auxiliary lemma involving two matrices, namely $\Theta = \Theta_s \in \mathbb{R}^{2^s \times 2^s}$ and $\Theta' = \Theta'_s \in \mathbb{R}^{(2^s-1)\times(2^s-1)}$ defined as

$$\Theta_s := \left[(-1)^{\eta^T \theta} : \begin{matrix} \eta \in E_s \\ \theta \in E_s \end{matrix} \right], \qquad \Theta'_s := \left[(-1)^{\eta^T \theta} : \begin{matrix} \eta \in E'_s \\ \theta \in E'_s \end{matrix} \right], \tag{23}$$

and related each other by the fact that $\Theta = \begin{bmatrix} 1 & \mathbf{1}^T \\ \mathbf{1} & \Theta' \end{bmatrix}$, with η and θ being ordered lexicographically.

Lemma 7. *For any $s \in \mathbb{N}$ the matrices Θ_s and Θ'_s are nonsingular.*

Proof. Note that

$$\Theta_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \Theta_1^{-1} \quad \text{and} \quad \Theta'_1 = -1.$$

and

$$\Theta_{s+1} = \begin{bmatrix} \Theta_s & \Theta_s \\ \Theta_s & -\Theta_s \end{bmatrix} = \Theta_1 \otimes \Theta_s, \quad \text{hence } \Theta_s = \bigotimes_{j=1}^s \Theta_1, \tag{24}$$

so that $\det \Theta_{s+1} = (\det \Theta_1)^{2^s} (\det \Theta_s)^2 = 2^{(s+1)2^s}$, which is easily verified by induction.

Since Θ'_s is the lower right principal submatrix of Θ_s, its determinant is the upper left entry in the *adjoint* or *adjugate* of Θ_s, cf. [11, Theorem 6.4, p. 136], so that

$$1 = (\Theta_s)_{11} = 2^s \left(\Theta_s^{-1}\right)_{11} = 2^s \, (\det \Theta_s)^{-1} \det \Theta_s',$$

hence $\det \Theta_s' = 2^{-s} \det \Theta_s \neq 0$. □

Remark 8. The nonsingularity of Θ_s could also be proved in a different way. Using the formula

$$\sum_{\theta \in E_s} (-1)^{\eta^T \theta} = 2^s \, \delta_{\eta,0}, \quad \eta \in E_s, \tag{25}$$

well known from discrete Fourier transforms, it can be easily shown that $\Theta_s^2 = 2^s \boldsymbol{I}$, from which nonsingularity clearly follows as $\Theta_s^{-1} = 2^{-s} \Theta_s$.

Now we can characterize the reproduction of polynomial vector sequences by the interpolatory condition.

Theorem 9. *A full rank interpolatory subdivision scheme based on a mask $\boldsymbol{A} \in \ell_0^{r \times r}(\mathbb{Z}^s)$ reproduces all polynomials of degree at most n if and only if*

$$\sum_{\alpha \in \mathbb{Z}^s} (2\alpha + \eta)^\beta \boldsymbol{A}_{2\alpha+\eta} = \boldsymbol{0}, \quad |\beta| \le n, \eta \in E'. \tag{26}$$

Proof. Making use of a characterization from [6], and taking into account that the symbol is properly normalized at 1 because of the interpolatory conditions, we can equivalently describe polynomial preservation by

$$D^\beta \boldsymbol{A}^* \left((-1)^\theta\right) = \boldsymbol{0}, \quad |\beta| \le n, \theta \in E'. \tag{27}$$

We begin by computing for $\theta \in E'$

$$\begin{aligned} D^\beta \boldsymbol{A}^* \left((-1)^\theta\right) &= \sum_{\alpha \in \mathbb{Z}^s} \frac{\alpha!}{(\alpha-\beta)!} \boldsymbol{A}_\alpha \, (-1)^{\theta^T(\alpha-\beta)} \\ &= (-1)^{\theta^T \beta} \sum_{\alpha \in \mathbb{Z}^s} q_\beta(\alpha) \boldsymbol{A}_\alpha \, (-1)^{\theta^T \alpha}, \end{aligned}$$

where q_β is a monic polynomial of the form $q_\beta(x) = x^\beta + \widetilde{q}_\beta(x)$, $\deg \widetilde{q}_\beta < |\beta|$, so that an inductive argument exactly like the one in [4, Section 3] yields that (27) is equivalent to

$$\boldsymbol{0} = \sum_{\alpha \in \mathbb{Z}^s} \alpha^\beta \boldsymbol{A}_\alpha \, (-1)^{\theta^T \alpha}, \quad \theta \in E'. \tag{28}$$

Splitting the sum modulo E and taking into account that the terms with $\eta = 0$ all vanish because the scheme is interpolatory, we also obtain another equivalent description, namely

$$\boldsymbol{0} = \sum_{\eta \in E'} (-1)^{\theta^T \eta} \sum_{\alpha \in \mathbb{Z}^s} (2\alpha + \eta)^\beta \, \boldsymbol{A}_{2\alpha+\eta}, \quad \theta \in E'. \tag{29}$$

By Lemma 7 we know that for any $\eta \in E'$ there exist numbers $c_{\theta\eta} \in \mathbb{R}$, $\theta \in E'$ such that

$$\sum_{\theta \in E'} (-1)^{\zeta^T \theta} \, c_{\theta\eta} = 2^s \, \delta_{\zeta,\eta}, \quad \zeta \in E'.$$

Hence, if we multiply (29) by these numbers for $\theta \in E'$ and sum over θ, then we immediately get (26).

Conversely, we only need to substitute (26) into (29) to also obtain the equivalent (27), and therefore the polynomial reproduction. □

With a slightly different argument we can also show that certain symmetries can be used to improve the order of polynomial reproduction provided by a full rank interpolatory subdivision scheme.

We now return to (28) and pick an index $j \in \{1, \ldots, s\}$ such that $\beta_j \neq 0$. Then we can transform (28) into

$$\begin{aligned}\mathbf{0} &= \sum_{\widehat{\alpha}\in\mathbb{Z}_j^s}\sum_{\alpha_j\in\mathbb{N}} \left(\widehat{\alpha}+\alpha_j\,\epsilon_j\right)^{\beta}\left(\boldsymbol{A}_{\widehat{\alpha}+\alpha_j\,\epsilon_j}+(-1)^{\beta_j}\boldsymbol{A}_{\widehat{\alpha}-k_j\,\epsilon_j}\right)(-1)^{\theta^T(\widehat{\alpha}+\alpha_j\,\epsilon_j)}\\ &= \sum_{\alpha\in\mathbb{Z}_j^s+\mathbb{N}_{\{j\}^c}^s} \alpha^{\beta}\left(\boldsymbol{A}_{\alpha}+(-1)^{\beta_j}\boldsymbol{A}_{\alpha-2\alpha_j\,\epsilon_j}\right)(-1)^{\theta^T\alpha},\end{aligned}$$

and, more general, with any J such that $\beta \notin \mathbb{Z}_J^s$

$$\mathbf{0} = \sum_{\alpha\in\mathbb{Z}_J^s+\mathbb{N}_{J^c}^s} \alpha^{\beta}(-1)^{\theta^T\alpha}\sum_{\eta\in E_{J^c}}(-1)^{\eta^{\mathrm{T}}\beta}\boldsymbol{A}_{\alpha-2\eta\cdot\alpha}. \tag{30}$$

Expanding (30) as in the preceding proof and using the same argument as above, we can thus give the “symmetry” conditions for polynomial reproduction.

Theorem 10. *The full rank interpolatory subdivision scheme reproduces polynomials of degree at most n if and only if for any $|\beta| \le n$ and any J such that $\beta \notin \mathbb{Z}_J^s$ one has*

$$\mathbf{0} = \sum_{\alpha\in\mathbb{Z}_J^s+\mathbb{N}_{J^c}^s} (2\alpha+\eta)^{\beta}\sum_{\theta\in E_{J^c}}(-1)^{\theta^T\beta}\boldsymbol{A}_{2\alpha+\eta-2\theta\cdot(2\alpha+\eta)}, \qquad \eta\in E'. \tag{31}$$

Corollary 11. *If $\boldsymbol{A}$ satisfies for all $|\beta| \le n$ and an appropriate $J = J_\beta$ such that $\beta \notin \mathbb{Z}_J^s$, the symmetry condition*

$$\sum_{\theta\in E_{J^c}}(-1)^{\theta^T\beta}\boldsymbol{A}_{2\alpha+\eta-2\theta\cdot(2\alpha+\eta)} = \mathbf{0}, \qquad \alpha\in\mathbb{Z}_J^s+\mathbb{N}_{J^c}^s,\ \eta\in E', \tag{32}$$

then $S_{\boldsymbol{A}}$ reproduces all polynomials of degree at most n.

Normally, Corollary 11 is not a reasonable criterion since the conditions are very complicated and restrictive. However, one interesting application is that any *axially symmetric* full rank polynomial subdivision scheme already preserves linear polynomials.

Corollary 12. *If $\boldsymbol{A}_\alpha = \boldsymbol{A}_{\alpha-2\alpha_j\,\epsilon_j}$ for any $\alpha\in\mathbb{Z}^s$ and $j = 1, \ldots, s$, then $S_{\boldsymbol{A}}$ preserves linear polynomials.*

Proof. Consider (32) for $\beta = \epsilon_j$ and the associated $J = \{j\}$, $j = 1, \ldots, s$. □

6. The matrix tensor product case

The simplest way to construct a multivariate full rank interpolatory subdivision scheme is by means of tensor product. In the scalar situation, given s univariate scalar interpolatory masks, a^j,

$j = 1, \ldots, s$, satisfying

$$\left(a^j\right)^* (1) = 2, \qquad \left(a^j\right)^* (-1) = 0, \qquad \left(a^j\right)^* (z) + \left(a^j\right)^* (-z) = 2, \quad j = 1, \ldots, s,$$

the associated multivariate symbol

$$a^*(z) = \prod_{j=1}^{s} \left(a^j\right)^* \left(z_j\right)$$

certainly satisfies the interpolatory conditions

$$a^*(1) = 2^s, \qquad a^*\left((-1)^\eta\right) = 0, \quad \eta \in E, \qquad \sum_{\eta \in E} a^*\left((-1)^\eta \cdot z\right) = 2^s$$

from (13). The matrix counterpart in a tensor product construction is again the *Kronecker product* of matrix symbols of even different sizes. Given s univariate matrix interpolatory subdivision masks $\boldsymbol{A}^j \in \ell_0^{r_j \times r_j}(\mathbb{Z})$, $j = 1, \ldots, s$, whose symbols satisfy

$$\left(\boldsymbol{A}^j\right)^* (1) = 2\boldsymbol{I}, \qquad \left(\boldsymbol{A}^j\right)^* (-1) = \boldsymbol{0}, \qquad \left(\boldsymbol{A}^j\right)^* (z) + \left(\boldsymbol{A}^j\right)^* (-z) = 2\boldsymbol{I}, \quad z \in \mathbb{C}_\times,$$

the tensor product matrix subdivision scheme $\boldsymbol{A} := \boldsymbol{A}^1 \otimes \cdots \otimes \boldsymbol{A}^s \in \ell_0^{r \times r}(\mathbb{R}^s)$, $r = r_1 \cdots r_s$, is then defined by means of its symbol as

$$\boldsymbol{A}^*(z) := \bigotimes_{j=1}^{s} \left(\boldsymbol{A}^j\right)^* \left(z_j\right) = \left(\boldsymbol{A}^1\right)^* (z_1) \otimes \cdots \otimes \left(\boldsymbol{A}^s\right)^* (z_s). \tag{33}$$

A matrix subdivision scheme is called *separable* if it can be written as a tensor product of the form (33). Since the Kronecker product of identity matrices and of zero matrices is again the identity and the zero matrix, respectively, and the Kronecker product of matrices is distributive with respect to the sum of matrices, it is not difficult to check that $\boldsymbol{A}^*(z)$ is an interpolatory full rank scheme since it satisfies

$$\boldsymbol{A}^*(1) = 2^s \boldsymbol{I}, \qquad \boldsymbol{A}^*\left((-1)^\eta\right) = \boldsymbol{0}, \quad \eta \in E', \qquad \sum_{\eta \in E} \boldsymbol{A}^*\left((-1)^\eta \cdot z\right) = 2^s \boldsymbol{I}.$$

Concerning the convergence of the multivariate separable symbol $\boldsymbol{A}^*(z)$, we can prove the following result.

Proposition 13. *Let $\boldsymbol{A}^j$ be a univariate matrix subdivision scheme with associated basic limit function $\boldsymbol{F}^j$, $j = 1, \ldots, s$. Then the tensor product subdivision scheme $\boldsymbol{A} = \boldsymbol{A}^1 \otimes \cdots \otimes \boldsymbol{A}^s$ converges with basic limit function $\boldsymbol{F} = \boldsymbol{F}^1(x_1) \otimes \cdots \otimes \boldsymbol{F}^s(x_s)$.*

Proof. Since $\boldsymbol{A}^*(z) = \left(\boldsymbol{A}^1\right)^* (z_1) \otimes \cdots \otimes (\boldsymbol{A}^s)^* (z_s)$ is equivalent to

$$\boldsymbol{A}_\alpha = \bigotimes_{j=1}^{s} \boldsymbol{A}^j_{\alpha_j}, \quad \alpha \in \mathbb{Z}^s, \tag{34}$$

we get for the iterated mask $\boldsymbol{A}^{(k)} = S_{\boldsymbol{A}}^k \delta \boldsymbol{I}$, $k \geq 1$, satisfying

$$\boldsymbol{A}^{(1)}_\alpha = \boldsymbol{A}_\alpha, \qquad \boldsymbol{A}^{(k)}_\alpha = \sum_{\beta \in \mathbb{Z}^s} \boldsymbol{A}_{\alpha - 2\beta} \boldsymbol{A}^{(k-1)}_\beta, \quad \alpha \in \mathbb{Z}^s,$$

that

$$\boldsymbol{A}_\alpha^{(k)} = \bigotimes_{j=1}^{s} \left(\boldsymbol{A}^j\right)_{\alpha_j}^{(k)}.$$

This is easily proved by induction noting that the case $k = 1$ is (34), while the inductive step is obtained by means of the well-known properties of the Kronecker product:

$$\begin{aligned}
\boldsymbol{A}_\alpha^{(k)} &= \sum_{\beta\in\mathbb{Z}^s} \left(\bigotimes_{j=1}^{s} \boldsymbol{A}_{\alpha_j-2\beta_j}^{j}\right)\left(\bigotimes_{j=1}^{s} \left(\boldsymbol{A}^j\right)_{\beta_j}^{(k-1)}\right)\\
&= \sum_{\beta\in\mathbb{Z}^s} \bigotimes_{j=1}^{s} \left(\boldsymbol{A}_{\alpha_j-2\beta_j}^{j} \left(\boldsymbol{A}^j\right)_{\beta_j}^{(k-1)}\right)\\
&= \bigotimes_{j=1}^{s} \sum_{\ell\in\mathbb{Z}} \boldsymbol{A}_{\alpha_j-2\ell}^{j} \left(\boldsymbol{A}^j\right)_{\ell}^{(k-1)} = \bigotimes_{j=1}^{s} \left(\boldsymbol{A}^j\right)_{\alpha_j}^{(k)}.
\end{aligned}$$

Consequently, with $\boldsymbol{F}(x) = \bigotimes_{j=1}^{s} \boldsymbol{F}^j\left(x_j\right)$, we have for $\alpha \in \mathbb{Z}^s$ that

$$\begin{aligned}
\boldsymbol{F}\left(\frac{\alpha}{2^k}\right) - \boldsymbol{A}_\alpha^{(k)} &= \bigotimes_{j=1}^{s} \boldsymbol{F}^j\left(\frac{\alpha_j}{2^k}\right) - \bigotimes_{j=1}^{s} \left(\boldsymbol{A}^j\right)_{\alpha_j}^{(k)}\\
&= \sum_{\ell=1}^{s} \left(\bigotimes_{j=1}^{\ell-1} \boldsymbol{F}^j\left(\frac{\alpha_j}{2^k}\right)\right) \otimes \left(\boldsymbol{F}^\ell\left(\frac{\alpha_\ell}{2^k}\right) - \left(\boldsymbol{A}^\ell\right)_{\alpha_\ell}^{(k)}\right) \otimes \bigotimes_{j=\ell+1}^{s} \left(\boldsymbol{A}^j\right)_{\alpha_j}^{(k)}. \quad (35)
\end{aligned}$$

Since the norm of a tensor product is bounded by the product of the norms,

$$|\boldsymbol{A}|_\infty \le \prod_{j=1}^{s} \left|\boldsymbol{A}^j\right|_\infty, \qquad \sup_{x\in\mathbb{R}^s} |\boldsymbol{F}(x)|_\infty \le \prod_{j=1}^{s} \sup_{x_j\in\mathbb{R}} \left|\boldsymbol{F}^j\left(x_j\right)\right|_\infty,$$

the claim follows from (35) and the convergence assumptions

$$\lim_{k\to\infty} \max_{j\in\{1,\dots,s\}} \sup_{\alpha_j\in\mathbb{Z}} \left|\boldsymbol{F}^j\left(\frac{\alpha_j}{2^k}\right) - \left(\boldsymbol{A}^j\right)_{\alpha_j}^{(k)}\right|_\infty = 0. \quad \square$$

Since $D^\beta \boldsymbol{A}^*(z) = \bigotimes_{j=1}^{s} D^{\beta_j} \left(\boldsymbol{A}^j\right)^* \left(z_j\right)$, the degree of polynomial reproduction of such a tensor product scheme is an immediate consequence of Theorem 2.

Proposition 14. *Let $\boldsymbol{A}^j$, $j = 1, \dots, s$, be s univariate matrix subdivision schemes of matrix dimension r_j, reproducing polynomials up to degree k_j, $j = 1, \dots, s$. Then the tensor product subdivision scheme $\boldsymbol{A} = \boldsymbol{A}^1 \otimes \cdots \otimes \boldsymbol{A}^s$ reproduces*

$$\bigotimes_{j=1}^{s} \Pi_{k_j}^{r_j} = \left\{\bigotimes_{j=1}^{s} \boldsymbol{p}_j : \boldsymbol{p}_j \in \Pi_{k_j}^{r_j}\right\}.$$

The tensor product polynomial space $\bigotimes_{j=1}^{s} \Pi_{k_j}^{r_j}$ is a subspace of

$$\Pi_\kappa^r = \left\{\boldsymbol{p}(z) = \sum_{0\le\alpha\le\kappa} \boldsymbol{p}_\alpha\, z^\alpha, \boldsymbol{p}_\alpha \in \mathbb{R}^r\right\}, \quad r = r_1 \dots r_s,$$

the space of all polynomials of *coordinate degree* at most $\kappa = (k_1, \ldots, k_s)$. However, it is a proper subspace, as the example of the constant polynomial $[1, 0, 0, 1]^T \in \Pi^4_{(0,0)} \setminus \left(\Pi^2_0 \otimes \Pi^2_0\right)$ shows. Indeed, since any polynomial in $\Pi^2_0 \otimes \Pi^2_0$ has to be of the form

$$\boldsymbol{p}(z_1, z_2) = \boldsymbol{p}(z_1) \otimes \boldsymbol{q}(z_2) = \begin{bmatrix} p_1(z_1)\, q_1(z_2) \\ p_1(z_1)\, q_2(z_2) \\ p_2(z_1)\, q_1(z_2) \\ p_2(z_1)\, q_2(z_2) \end{bmatrix},$$

the first two conditions of $\boldsymbol{p} = [1, 0, 0, 1]^T$ would imply $p_1 = q_1 = 1$ and then $q_2 = 0$ while the last one leads to the contradictory $p_2 = q_2 = 1$.

Lemma 15. *For $r_j \in \mathbb{N}$, $j = 1, \ldots, s$, set*

$$\mathbb{R}^\rho := \mathbb{R}^{(r_1,\ldots,r_s)} = \bigotimes_{j=1}^{s} \mathbb{R}^{r_j} = \left\{ \bigotimes_{j=1}^{s} \boldsymbol{y}_j : \boldsymbol{y}_j \in \mathbb{R}^{r_j} \right\}.$$

Then for $\kappa = (k_1, \ldots, k_s)$,

$$\Pi^\rho_\kappa := \bigotimes_{j=1}^{s} \Pi^{r_j}_{k_j} = \left\{ \sum_{0 \le \alpha \le \kappa} \boldsymbol{p}_\alpha z^\alpha : \boldsymbol{p}_\alpha \in \mathbb{R}^\rho \right\}.$$

Proof. As in the proof of Proposition 13 we use the distributive property of the Kronecker product to obtain that

$$\bigotimes_{j=1}^{s} \boldsymbol{p}_j\left(z_j\right) = \bigotimes_{j=1}^{s} \left(\sum_{k=0}^{k_j} \boldsymbol{p}_{j,k}\, z_j^k \right) = \sum_{0 \le \alpha \le \kappa} \left(\bigotimes_{j=1}^{s} \boldsymbol{p}_{j,\alpha_j} \right) z^\alpha. \quad \square$$

An obvious “disadvantage” of the Kronecker approach is the increase of the matrix size. As long as one is only interested to set up tensor product matrix subdivision schemes in several variables, this can be avoided by taking the Kronecker product of $s - 1$ scalar univariate scheme and only one matrix scheme. In view of Lemma 15 this also has the advantage that the resulting matrix subdivision really reproduces all vector valued tensor product polynomials up to a certain coordinate degree.

7. Examples of full rank interpolatory schemes

7.1. A bivariate tensor product example

Let us consider the univariate symbols

$$a_1^*(z_1) = \frac{(z_1+1)^2}{2z_1}, \qquad \boldsymbol{A}_2^*(z_2) = \frac{1}{2} \begin{pmatrix} \dfrac{(z_2+1)^2}{z_2} & \dfrac{(z_2^2-1)^3 z_2}{10} \\ \dfrac{(z_2^{-2}-1)^3}{10 z_2} & -\dfrac{(z_2^2-4z_2+1)(z_2+1)^4}{8\, z_2^3} \end{pmatrix}$$

to construct the bivariate symbol $\boldsymbol{A}^*(z_1, z_2) = a_1^*(z_1) \otimes \boldsymbol{A}_2^*(z_2)$ given by

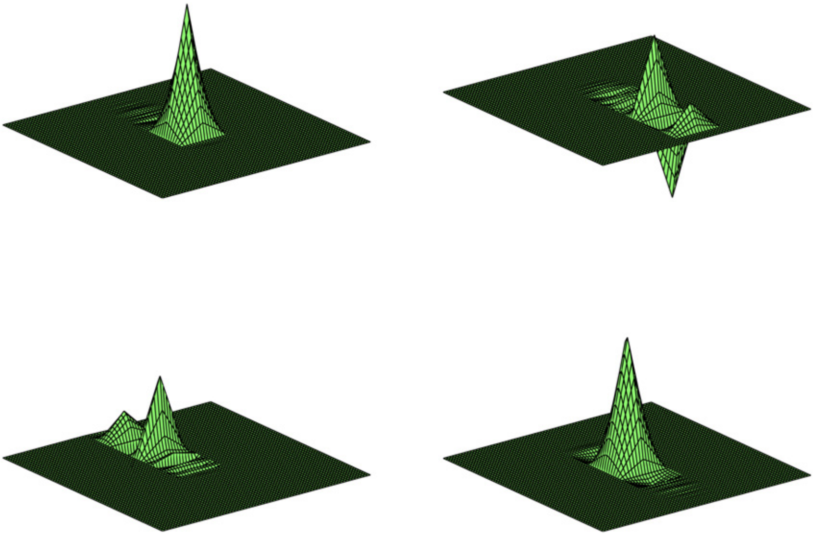

Fig. 1. Plot of $S_{\boldsymbol{A}}^3 \delta \boldsymbol{I}$ for $\boldsymbol{A}$ as in (36).

$$\boldsymbol{A}^*(z_1, z_2) = \frac{1}{4}\begin{pmatrix} \dfrac{(z_1+1)^2(z_2+1)^2}{z_1 z_2} & \dfrac{(z_1+1)^2(z_2^2-1)^3 z_2}{10\, z_1} \\ \dfrac{(z_1+1)^2(z_2^{-2}-1)^3}{10\, z_1 z_2} & -\dfrac{(z_1+1)^2(z_2^2-4z_2+1)(z_2+1)^4}{8\, z_1 z_2^3} \end{pmatrix}. \tag{36}$$

Since the subdivision scheme associated with $a_1^*(z_1)$ is known to be convergent to the linear spline interpolant, convergence of the bivariate full rank interpolatory subdivision scheme associated with $\boldsymbol{A}^*(z_1, z_2)$, can, in view of Proposition 13, be verified by proving convergence of the scheme associate with $\boldsymbol{A}_2^*(z_2)$. To this purpose we write it as

$$\boldsymbol{A}_2^*(z_2) = (z_2+1)\boldsymbol{B}_2^*(z_2),$$

with

$$\boldsymbol{B}_2^*(z_2) := (z_2+1)\begin{pmatrix} \dfrac{z_2+1}{2z_2} & \dfrac{(z_2-1)^3(z_2+1)^2 z_2}{20} \\ \dfrac{(z_2^{-1}-1)^3(z_2^{-1}+1)^2}{20 z_2^2} & -\dfrac{(z_2^2-4z_2+1)(z_2+1)^3}{16\, z_2^3} \end{pmatrix}$$

and note that $\boldsymbol{B}_2^*(z_2)$ is even *level* 1 *contractive*, that is $\|S_{\boldsymbol{B}_2}\|_\infty < 1$, a sufficient condition for the scheme with symbol $\boldsymbol{A}_2^*(z)$ to be convergent, cf. [9].

The plot of three steps of the subdivision scheme $S_{\boldsymbol{A}}$ when staring with the delta sequence, $\delta\boldsymbol{I}$, is shown in Fig. 1.

7.2. A bivariate non-tensor product example

We consider the following interpolatory full rank symbol:

$$\boldsymbol{A}^*(z_1, z_2) = \begin{bmatrix} a_{11}^*(z_1, z_2) & a_{12}^*(z_1, z_2) \\ a_{12}^*(z_1, z_2) & a_{22}^*(z_1, z_2) \end{bmatrix} \tag{37}$$

where

$$a_{11}^*(z_1,z_2) = \frac{1}{4}\begin{bmatrix}1 & z_1 & z_1^2\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}1\\ z_2\\ z_2^2\end{bmatrix} = \frac{1}{4}\,(z_2+1)^2\,(z_1+1)^2$$

is the symbol of a bilinear B-spline and

$$a_{22}^*(z_1,z_2) = \frac{1}{2}\begin{bmatrix}1 & z_1 & z_1^2\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\ z_2\\ z_2^2\end{bmatrix} = \frac{1}{2}\,(z_2+1)\,(z_1+1)\,(1+z_1z_2)$$

is the symbol of the three direction box spline $B_{1,1,1}$. Note that both $a_{11}^*(z_1,z_2)$ and $a_{22}^*(z_1,z_2)$ are scalar interpolatory symbols, which is a necessary requirement. For the off-diagonal elements, according to Proposition 6, we chose

$$a_{12}^*(z_1,z_2) = \lambda(z_1^2-1)(z_2^{-1}+z_2) + \mu(z_2^2-1)(z_1^{-1}+z_1)$$

with parameters $\lambda, \mu \in \mathbb{R}_+$. To check convergence of the subdivision scheme $S_{\boldsymbol{A}}$ we derive a difference mask $\boldsymbol{B} \in \ell_0^{4\times 4}(\mathbb{Z}^2)$ and set λ and μ such that $\|S_{\boldsymbol{B}}\| < 1$.

It is not difficult to see that for $\nabla(z) := \begin{pmatrix} z_1-1 & 0\\ 0 & z_1-1\\ z_2-1 & 0\\ 0 & z_2-1\end{pmatrix}$ we have

$$\nabla(z)\boldsymbol{A}^*(z_1,z_2) = \begin{pmatrix}\boldsymbol{B}_{11}^*(z_1,z_2) & \boldsymbol{B}_{12}^*(z_1,z_2)\\ \boldsymbol{B}_{21}^*(z_1,z_2) & \boldsymbol{B}_{22}^*(z_1,z_2)\end{pmatrix}\nabla(z^2)$$

where

$$\boldsymbol{B}_{11}^*(z_1,z_2) := \begin{pmatrix}\frac{1}{4}\,(z_1+1)\,(z_2+1)^2 & \lambda(z_1-1)(z_2^{-1}+z_2)\\ \lambda(z_1-1)(z_2^{-1}+z_2) & \frac{1}{2}\,(z_2+1)\,(1+z_1z_2)\end{pmatrix},$$

$$\boldsymbol{B}_{12}^*(z_1,z_2) := \begin{pmatrix}0 & \mu(z_1-1)(z_1^{-1}+z_1)\\ \mu(z_1-1)(z_1^{-1}+z_1) & 0\end{pmatrix},$$

$$\boldsymbol{B}_{21}^*(z_1,z_2) := \begin{pmatrix}0 & \lambda(z_2-1)(z_2^{-1}+z_2)\\ \lambda(z_2-1)(z_2^{-1}+z_2) & 0\end{pmatrix},$$

and

$$\boldsymbol{B}_{22}^*(z_1,z_2) := \begin{pmatrix}\frac{1}{4}\,(z_1+1)^2\,(z_2+1) & \mu(z_2-1)(z_1^{-1}+z_1)\\ \mu(z_2-1)(z_1^{-1}+z_1) & \frac{1}{2}\,(z_1+1)\,(1+z_1z_2)\end{pmatrix}.$$

Since $\|S_{\boldsymbol{B}}\|_\infty := \max_{\eta\in E}\left|\sum_{\beta\in\mathbb{Z}^s}|\boldsymbol{B}_{\eta-2\beta}|\right|_\infty = \max\left\{\frac{1}{2}+2\lambda,\ \frac{1}{2}+2\mu\right\}$, any choice of λ and μ such that $\max\{\lambda,\ \mu\} < \frac{1}{4}$ guarantees the convergence of the subdivision scheme $S_{\boldsymbol{A}}$.

The plot of 3 steps of the subdivision scheme associated to the symbol $\boldsymbol{A}^*(z)$ for the special values of $\lambda = \mu = 1/10$ and applied to the initial delta sequence, $\delta\boldsymbol{I}$, is shown in Fig. 2.

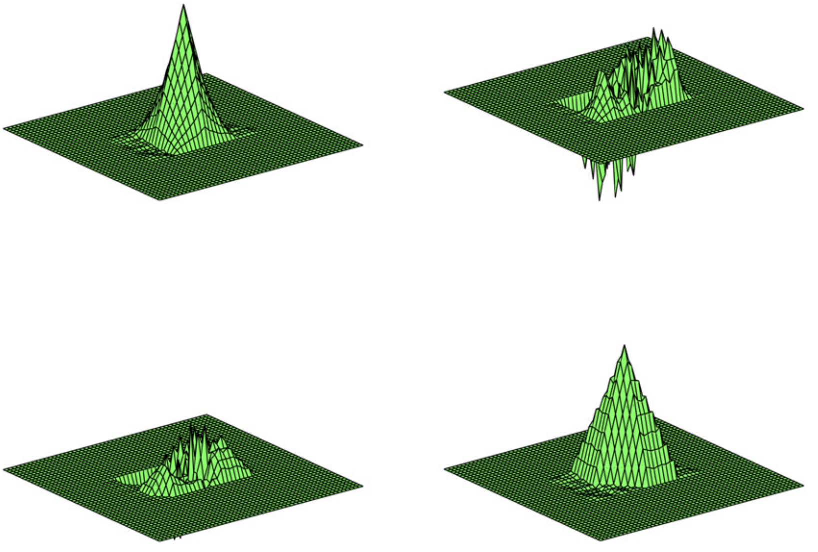

Fig. 2. Plot of $S_{\boldsymbol{A}}^3 \delta \boldsymbol{I}$ for $\boldsymbol{A}$ as in (37).

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